

# K-THEORY OF $C^*$ -ALGEBRAS FROM ONE-DIMENSIONAL GENERALIZED SOLENOIDS

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ABSTRACT. We compute the  $K$ -groups of  $C^*$ -algebras arising from one-dimensional generalized solenoids. The results show that Ruelle algebras from one-dimensional generalized solenoids are one-dimensional generalizations of Cuntz-Krieger algebras.

## 1. INTRODUCTION

Ian Putnam and David Ruelle have developed a theory of  $C^*$ -algebras for certain hyperbolic dynamical systems ([16, 17, 18, 21]). These systems include Anosov diffeomorphisms, topological Markov chains and some examples of substitution tiling systems. The corresponding  $C^*$ -algebras are modeled as reduced groupoid  $C^*$ -algebras for various equivalence relations.

This paper is concerned with  $C^*$ -algebras of an orientable one-dimensional generalized solenoid  $(\overline{X}, \overline{f})$ , where  $\overline{X}$  has local canonical coordinates which are contracting and expanding directions for  $\overline{f}$ . Naïvely speaking, Williams's orientable generalized solenoids are higher dimensional analogues of topological Markov chains ([23, 24]). We consider the principal groupoids of stable and unstable equivalence on  $(\overline{X}, \overline{f})$ , denoted  $G_s(\overline{X}, \overline{f})$  and  $G_u(\overline{X}, \overline{f})$ , respectively. We give them topologies and Haar systems ([16, 17]) so that we may build their reduced groupoid  $C^*$ -algebras  $S(\overline{X}, \overline{f})$  and  $U(\overline{X}, \overline{f})$ , respectively, as in [19]. The homeomorphism  $\overline{f}: \overline{X} \rightarrow \overline{X}$  induces automorphism of  $G_s(\overline{X}, \overline{f})$  and  $G_u(\overline{X}, \overline{f})$ , and we form semi-direct products  $G_s \rtimes \mathbb{Z}$  and  $G_u \rtimes \mathbb{Z}$ . Their groupoid  $C^*$ -algebras are denoted  $R_s(\overline{X}, \overline{f})$  and  $R_u(\overline{X}, \overline{f})$ , respectively, and are called the *Ruelle algebras* ([17, 18]). In the case of topological Markov chains, the Ruelle algebras are the Cuntz-Krieger algebras, and the stable and unstable equivalence algebras are the corresponding  $AF$ -subalgebras of the Cuntz-Krieger algebras.

An important tool in the study of  $C^*$ -algebras is  $K$ -theory. Giordano, Herman, Putnam and Skau showed that almost complete information about the orbit structure of Cantor systems is encoded by the  $K$ -theory of their associated  $C^*$ -algebras ([5, 6]). And Kirchberg and Phillips showed in their recent papers ([8, 14]) that nuclear, purely infinite, separable, simple  $C^*$ -algebras are classified by their  $K$ -theory.

In this paper, we compute the  $K$ -groups of the unstable equivalence algebras and the Ruelle algebras of 1-solenoids to answer the questions posed in [17, §4]. We show that the unstable equivalence algebra of a 1-solenoid  $(\overline{X}, \overline{f})$  with an adjacency matrix  $M$  is strongly Morita equivalent to the crossed product of a natural Cantor system of  $(\overline{X}, \overline{f})$  by  $\mathbb{Z}$  so that its  $K_0$ -group is order isomorphic to the dimension group of  $M$  and its  $K_1$ -group is  $\mathbb{Z}$ . Then we use the Pimsner-Voiculescu exact

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sequence, the Universal Coefficient Theorem and Spanier-Whitehead duality to obtain that the  $K_0$ -groups of Ruelle algebras are isomorphic to  $\mathbb{Z} \oplus \{\Delta_M / \text{Im}(Id - \delta_M)\}$  and the  $K_1$ -groups are  $\mathbb{Z} \oplus \text{Ker}(Id - \delta_M)$ . Thus  $C^*$ -algebras from one-dimensional generalized solenoids are one-dimensional analogues of the Cuntz-Krieger algebras.

The outline of the paper is as follow: In section 2, we recall the axioms of one-dimensional generalized solenoids and their ordered group invariants. In section 3, we review the definitions of Smale spaces, and show that orientable one-dimensional solenoids are Smale spaces. Then we observe that the stable equivalence algebras are strongly Morita equivalent to inductive limit systems of  $C^*$ -algebras, and that the  $K$ -theory of the unstable equivalence algebras are determined by the adjacency matrices of one-dimensional generalized solenoids. In section 4, we compute  $K$ -groups of unstable and stable Ruelle algebras, and show that they are  $*$ -isomorphic to each other by the classification theorem of Kirchberg-Phillips.

## 2. ONE-DIMENSIONAL SOLENOIDS

We review the properties of one-dimensional generalized solenoids of Williams which will be used in later sections. As general references for the notions of one-dimensional generalized solenoids and their ordered group invariants we refer to [23, 24, 25, 26].

**One-dimensional generalized solenoids.** Let  $X$  be a finite directed graph with vertex set  $\mathcal{V}$  and edge set  $\mathcal{E}$ , and  $f: X \rightarrow X$  a continuous map. We define some axioms which might be satisfied by  $(X, f)$  ([25]).

- Axiom 0. (*Indecomposability*)  $(X, f)$  is indecomposable.
- Axiom 1. (*Nonwandering*) All points of  $X$  are nonwandering under  $f$ .
- Axiom 2. (*Flattening*) There is  $k \geq 1$  such that for all  $x \in X$  there is an open neighborhood  $U$  of  $x$  such that  $f^k(U)$  is homeomorphic to  $(-\epsilon, \epsilon)$ .
- Axiom 3. (*Expansion*) There are a metric  $d$  compatible with the topology and positive constants  $C$  and  $\lambda$  with  $\lambda > 1$  such that for all  $n > 0$  and all points  $x, y$  on a common edge of  $X$ , if  $f^n$  maps the interval  $[x, y]$  into an edge, then  $d(f^n x, f^n y) \geq C\lambda^n d(x, y)$ .
- Axiom 4. (*Nonfolding*)  $f^n|_{X-\mathcal{V}}$  is locally one-to-one for every positive integer  $n$ .
- Axiom 5. (*Markov*)  $f(\mathcal{V}) \subseteq \mathcal{V}$ .

Let  $\overline{X}$  be the inverse limit space

$$\overline{X} = X \xleftarrow{f} X \xleftarrow{f} \cdots = \{(x_0, x_1, x_2, \dots) \in \prod_0^\infty X \mid f(x_{n+1}) = x_n\},$$

and  $\overline{f}: \overline{X} \rightarrow \overline{X}$  the induced homeomorphism defined by

$$(x_0, x_1, x_2, \dots) \mapsto (f(x_0), f(x_1), f(x_2), \dots) = (f(x_0), x_0, x_1, \dots).$$

*Remark 2.1.* Williams' construction ([24, 6.2]) gives a (unique) measure  $\mu_0$  for which there is a constant  $\lambda > 1$  such that  $\mu_0(X) = 1$  and  $\mu_0(f(I)) = \lambda\mu_0(I)$  for every small interval  $I \subset X$ . Define  $d_0(x_0, y_0)$  to be the measure of the smallest interval from  $x_0$  to  $y_0$  in  $X$ , and

$$d(x, y) = \sum_{i=0}^{\infty} \lambda^{-i} d_0(x_i, y_i)$$

for  $x = (x_0, x_1, x_2, \dots)$  and  $y = (y_0, y_1, y_2, \dots)$  in  $\overline{X}$ . Then  $(\overline{X}, d)$  is a compact metric space.

Let  $Y$  be a topological space and  $g: Y \rightarrow Y$  a homeomorphism. We call  $Y$  a **1-dimensional generalized solenoid** or **1-solenoid** and  $g$  a **solenoid map** if there exist a directed graph  $X$  and a continuous map  $f: X \rightarrow X$  such that  $(X, f)$  satisfies all six Axioms and  $(\overline{X}, \overline{f})$  is topologically conjugate to  $(Y, g)$ . We call a point  $x \in X$  a *non-branch point* if  $x$  has an open neighborhood which is homeomorphic to an open interval, and **branch point** otherwise. An **elementary presentation**  $(X, f)$  of a 1-solenoid is such that  $X$  is a wedge of circles and  $f$  leaves the unique branch point of  $X$  fixed.

**Proposition 2.2** ([24, 5.2]). *For each 1-solenoid  $(\overline{X}, \overline{f})$ , there exists an integer  $m$  such that  $(\overline{X}, \overline{f^m})$  has an elementary presentation.*

Suppose that  $(X, f)$  is a presentation of a 1-solenoid. Since the inverse limit spaces of  $(X, f)$  and  $(X, f^n)$  are homeomorphic ([4]) for every positive integer  $n$ , for the purpose of computing invariants of the space  $\overline{X}$  there is no loss of generality in replacing  $(X, f)$  with  $(X, f^n)$  where  $n = m \cdot k$  is a positive integer such that  $(\overline{X}, \overline{f^m})$  has an elementary presentation  $(Y, g)$  and for every  $y \in Y$  there is an open set  $U_y$  such that  $g^k(U_y)$  is an open interval. Hence we can assume that every point  $x \in X$  has a neighborhood  $U_x$  such that  $f(U_x)$  is an interval.

Recall that a continuous map  $\gamma: [0, 1] \rightarrow G$ , a directed graph, is *orientation preserving* if  $e^{-1} \circ \gamma: I \rightarrow [0, 1]$  is increasing for every interval  $I \subset [0, 1]$  such that  $\gamma(I)$  is a subset of a directed edge  $e$ . A continuous map  $\phi: G_1 \rightarrow G_2$  between two directed graphs is *orientation preserving* if, for every orientation preserving map  $p: [0, 1] \rightarrow G_1$ , the map  $\phi \circ p: [0, 1] \rightarrow G_2$  is orientation preserving ([4]).

When we can give a direction to each edge of  $X$  so that the connection map  $f: X \rightarrow X$  is orientation preserving, we call  $(X, f)$  an **orientable presentation**. For a 1-solenoid  $Y$  with a solenoid map  $g$ , if there exists an orientable presentation  $(X, f)$  then  $Y$  is called an **orientable 1-solenoid**.

**Standing Assumption.** In this paper, we always assume that  $(X, f)$  is an orientable elementary presentation such that every point  $x \in X$  has a neighborhood  $U_x$  such that  $f(U_x)$  is an interval.

**Notation 2.3.** Suppose that  $(X, f)$  is a presentation of a 1-solenoid, and that  $\mathcal{E} = \{e_1, \dots, e_n\}$  is the edge set of the directed graph  $X$ . For each edge  $e_i \in \mathcal{E}$ , we can give  $e_i$  the *partition*  $\{I_{i,j}\}$ ,  $1 \leq j \leq l(i)$ , such that

- (1) the initial point of  $I_{i,1}$  is the initial point of  $e_i$ ,
- (2) the terminal point of  $I_{i,j}$  is the initial point of  $I_{i,j+1}$  for  $1 \leq j < l(i)$ ,
- (3) the terminal point of  $I_{i,l(i)}$  is the terminal point of  $e_i$ ,
- (4)  $f|_{\text{Int} I_{i,j}}$  is injective, and
- (5)  $f(I_{i,j}) = e_{i,j}^{s(i,j)}$  where  $e_{i,j} \in \mathcal{E}$ ,  $s(i,j) = 1$  if the direction of  $f(I_{i,j})$  agree with that of  $e_{i,j}$ , and  $s(i,j) = -1$  if the direction of  $f(I_{i,j})$  is reverse to that of  $e_{i,j}$ .

The *wrapping rule*  $\check{f}: \mathcal{E} \rightarrow \mathcal{E}^*$  associated with  $f$  is given by

$$\check{f}: e_i \mapsto e_{i,1}^{s(i,1)} \dots e_{i,l(i)}^{s(i,l(i))},$$

and the *adjacency matrix*  $M$  of  $(\mathcal{E}, \check{f})$  is given by

$$M(i, k) = \#\{I_{i,j} \mid f(I_{i,j}) = e_k^{\pm 1}\}.$$

*Remark 2.4* ([24, 6.2]). The measure  $\mu_0$  in remark 2.1 is given as follows: Suppose that  $\lambda$  is the Perron-Frobenius eigenvalue of the adjacency matrix  $M$  and that  $\mathbf{v} = (v_1, \dots, v_n)$  is the corresponding Perron eigenvector such that  $\sum_{i=1}^n v_i = 1$ . For edges  $e_i, e_j$  of  $X$  and an interval  $I$  of  $e_i$  such that  $f^n(I) = e_j$  and  $f^n|_{\text{Int}I}$  is injective, let

$$\mu_0(e_i) = v_i \text{ and } \mu_0(I) = \lambda^{-n} v_j.$$

Then  $\mu_0$  is extended to a regular Borel measure on  $X$  by the standard procedure.

**Theorem 2.5** ([1, 11, 27]). *Suppose that  $(\overline{X}, \overline{f})$  is a 1-solenoid. Then there exists a uniquely ergodic flow  $\phi$  whose phase space is  $\overline{X}$ .*

Suppose that  $(X, f)$  is a presentation of a 1-solenoid and that  $\mu_0$  is the measure given on  $X$  as in remark 2.4. For a measurable set  $I$  in  $X$ , we let  $U_n(I) = \{(x_0, \dots, x_n, \dots) \in \overline{X} \mid x_n \in I\}$ , and define a measure  $\mu$  on  $\overline{X}$  by

$$\mu(U_n(I)) = \mu_0(I).$$

Then  $\mu$  is extended to a regular Borel measure on  $\overline{X}$  in the standard way. We call this measure **Williams measure** of the flow  $\phi$  on  $\overline{X}$ . It is not difficult to verify that  $\mu$  is the unique  $\phi$ -invariant measure on  $\overline{X}$ .

A closed subset  $K$  of a phase space  $Y$  of a flow  $\phi$  is called a **cross section** if the mapping  $\phi: K \times \mathbb{R} \rightarrow Y$  defined by  $(p, t) \mapsto p \cdot t$  is a local homeomorphism onto  $Y$ . The **return time map**  $r_K: K \rightarrow K$  of a cross section  $K$  is defined by  $x \mapsto y = x \cdot t_x$  where  $x \in K$  and  $t_x$  is the smallest positive number such that  $x \cdot t_x = y \in K$ .

**Theorem 2.6** ([6, 26]). *Suppose that  $(\overline{X}, \overline{f})$  is a 1-solenoid with the corresponding adjacency matrix  $M$ , and that  $(K, r_K)$  is a cross section with the return time map of  $\overline{X}$ . Then*

- (1)  $K_1(C(K) \times_{r_K} \mathbb{Z}) = \mathbb{Z}$ ,
- (2)  $K_0(C(K) \times_{r_K} \mathbb{Z})$  is order isomorphic to  $\Delta_M$ , and
- (3)  $K_0(C(K) \times_{r_K} \mathbb{Z})$  has a unique state.

### 3. SMALE SPACES AND $C^*$ -ALGEBRAS FROM SOLENOIDS

**Smale spaces** ([16, 21]). Suppose that  $(Y, d)$  is a compact metric space and  $\varphi$  is a homeomorphism of  $Y$ . Assume that we have constants

$$0 < \lambda_0 < 1, \quad \epsilon_0 > 0$$

and a continuous map

$$(x, y) \in \{(x, y) \in Y \times Y \mid d(x, y) \leq 2\epsilon_0\} \mapsto [x, y] \in Y$$

satisfying the following:

$$[x, x] = x, \quad [[x, y], z] = [x, z], \quad [x, [y, z]] = [x, z], \quad [\varphi(x), \varphi(y)] = \varphi([x, y])$$

for  $x, y, z \in Y$  whenever both sides of the equation are defined. For every  $0 < \epsilon \leq \epsilon_0$  let

$$\begin{aligned} V^s(x, \epsilon) &= \{y \in Y \mid [x, y] = y \text{ and } d(x, y) < \epsilon\} \\ V^u(x, \epsilon) &= \{y \in Y \mid [y, x] = y \text{ and } d(x, y) < \epsilon\}. \end{aligned}$$

We assume that

$$\begin{aligned} d(\varphi(y), \varphi(z)) &\leq \lambda_0 d(y, z) \quad y, z \in V^s(x, \epsilon), \\ d(\varphi^{-1}(y), \varphi^{-1}(z)) &\leq \lambda_0 d(y, z) \quad y, z \in V^u(x, \epsilon). \end{aligned}$$

Then  $(Y, d, \varphi)$  is called a **Smale space**.

**Groupoids** ([17, 19]). We refer to the work of Renault ([19]) for the detailed theory of topological groupoids and their associated  $C^*$ -algebras. We give two examples of groupoids.

**Examples 3.1** ([21, 1.2]). (1) *Equivalence relations*. Suppose that  $R$  is an equivalence relation on a set  $S$ . We give  $R$  the following groupoid structure:

$$\begin{aligned} (s_1, t_1) \cdot (s_2, t_2) &= (s_1, t_2) \text{ if } t_1 = s_2 \text{ and} \\ (s, t)^{-1} &= (t, s). \end{aligned}$$

(2) *Flows*. Suppose that  $S$  is a zero dimensional space and  $r: S \rightarrow S$  is a homeomorphism. We consider the space  $S \times \mathbb{R}$  with the equivalence relation,  $(s, \tau + 1) \sim (r(s), \tau)$ . Let  $\Sigma = S \times \mathbb{R} / \sim$  be the quotient space and define a flow  $\phi: \Sigma \times \mathbb{R} \rightarrow \Sigma$  by  $\phi_t(s, \tau) = [(s, t + \tau)]$ . Give the following groupoid structure on  $\Sigma \times_\phi \mathbb{R}$ :

$$\begin{aligned} (\sigma_1, t_1) \cdot (\sigma_2, t_2) &= (\sigma_1, t_1 + t_2) \text{ if } \sigma_2 = \phi_{t_1}(\sigma_1) \text{ and} \\ (\sigma, t)^{-1} &= (\phi_t(\sigma), -t). \end{aligned}$$

For a Smale space  $(Y, d, \varphi)$ , define

$$G_{s,0} = \{(x, y) \in Y \times Y \mid y \in V^s(x, \epsilon_0)\} \quad G_{u,0} = \{(x, y) \in Y \times Y \mid y \in V^u(x, \epsilon_0)\}$$

and let

$$G_s = \bigcup_{n=0}^{\infty} (\varphi \times \varphi)^{-n}(G_{s,0}) \quad G_u = \bigcup_{n=0}^{\infty} (\varphi \times \varphi)^n(G_{u,0}).$$

Then  $G_s$  and  $G_u$  are equivalence relations on  $Y$ , called *stable* and *unstable* equivalence. Each  $(\varphi \times \varphi)^{-n}(G_{s,0})$ ,  $(\varphi \times \varphi)^n(G_{u,0})$  is given the relative topology of  $Y \times Y$ , and  $G_s$  and  $G_u$  are given the inductive limit topology. Then  $G_s$  and  $G_u$  are locally compact Hausdorff principal groupoids. The Haar systems  $\{\mu_s^x \mid x \in Y\}$  and  $\{\mu_u^x \mid x \in Y\}$  for  $G_s$  and  $G_u$ , respectively, are described in [17, 3.c]. The groupoid  $C^*$ -algebras of  $G_s$  and  $G_u$  are denoted  $S(Y, \varphi)$  and  $U(Y, \varphi)$ , respectively.

The map  $\varphi \times \varphi$  acts as an automorphism of  $G_s$  and  $G_u$ . We form the semi-direct products

$$\begin{aligned} G_s \rtimes \mathbb{Z} &= \{(x, n, y) \mid n \in \mathbb{Z} \text{ and } (\bar{f}^n(x), y) \in G_s\} \\ G_u \rtimes \mathbb{Z} &= \{(x, n, y) \mid n \in \mathbb{Z} \text{ and } (\bar{f}^n(x), y) \in G_u\} \end{aligned}$$

with groupoid operations

$$\begin{aligned} (x, n, y) \cdot (u, m, v) &= (x, n + m, v) \text{ if } y = u \text{ and} \\ (x, n, y)^{-1} &= (y, -n, x). \end{aligned}$$

The product topology of  $G_* \times \mathbb{Z}$  is transferred to  $G_* \rtimes \mathbb{Z}$  by the bijective map  $\eta: (x, y, n) \mapsto (x, n, \varphi(y))$ . And a Haar system on  $G_* \rtimes \mathbb{Z}$  is given by  $\mu_*^x \circ \eta^{-1}$  where  $\mu_*^x$  is the Haar system on  $G_*$ . The groupoid  $C^*$ -algebras  $C^*(G_s \rtimes \mathbb{Z})$  and  $C^*(G_u \rtimes \mathbb{Z})$  are denoted  $R_s(Y, \varphi)$  and  $R_u(Y, \varphi)$  and are called the *Ruelle algebras*.

**Theorem 3.2** ([7, 16, 17]). *Suppose that  $(Y, \varphi)$  is a topologically mixing Smale space. Then*

- (1)  *$S(Y, \varphi)$  and  $U(Y, \varphi)$  are amenable, nuclear, separable and simple  $C^*$ -algebras, and*
- (2)  *$R_s(Y, \varphi)$  and  $R_u(Y, \varphi)$  are amenable, non-unital, nuclear, purely infinite, separable, simple and stable  $C^*$ -algebras.*

For general properties of these  $C^*$ -algebras, we refer to [16, 17, 18].

Suppose that  $(\overline{X}, \overline{f})$  is a 1-solenoid with the metric  $d$  given in remark 2.1. Let  $\lambda_0 = \epsilon_0 = \frac{1}{\lambda}$  and define  $[\cdot, \cdot]: \overline{X} \times \overline{X} \rightarrow \overline{X}$  by  $[x, y] \mapsto z$  where  $z_0 = x_0$  and  $z_n$  is the unique element contained in the  $\lambda_0^{n+1}$ -neighborhood of  $y_n$  such that  $f^n(z_n) = x_0$ . Then it is not difficult to show that  $(\overline{X}, \overline{f}, d)$  satisfies the above conditions. Therefore we have the following:

**Proposition 3.3.** *One-dimensional generalized solenoids are Smale spaces.*

**Stable equivalence algebras for 1-solenoids.** Suppose that  $G_s$  is the stable equivalence groupoid of a 1-solenoid  $(\overline{X}, \overline{f})$  and that  $S(\overline{X}, \overline{f})$  is the corresponding groupoid algebra. We first repeat the structural question of Putnam ([17, §4]). For classical 1-solenoids, we refer to [3, 16].

**Question.** Can  $S(\overline{X}, \overline{f})$  be written as an inductive limit?

*Generalized transversals* ([18, §3]). Suppose that  $G_s$  is the stable equivalence groupoid of  $(\overline{X}, \overline{f})$ , that  $U_p$  is the unstable equivalence class of  $p \in \overline{X}$  with the inductive limit topology and that  $g: U_p \rightarrow G_{s,0}$  is given by  $x \mapsto (x, x)$  for  $x \in U_p$ . Let

$$G_s(p) = \{(x, y) \in G_s \mid x, y \in U_p\}.$$

A base for a topology on  $G_s(p)$  is

$$\{U \cap s^{-1} \circ g(V^s) \cap r^{-1} \circ g(V^r) \mid U \subset G_s, V^s, V^r \subset U_p \text{ are open sets}\}.$$

**Proposition 3.4** ([18, §3]). (1)  *$G_s(p)$  is an  $r$ -discrete, second countable, locally compact, Hausdorff groupoid with counting measure as Haar system.*  
 (2)  *$S(\overline{X}, \overline{f})$  is strongly Morita equivalent to  $C^*(G_s(p))$ .*

Now we choose  $p$  to be a fixed point of  $\overline{f}$  such that  $\pi_k(p)$  is contained in the interior of an edge  $e \in \mathcal{E}$ . Since the orbits of  $(\overline{X}, \mathbb{R}, \phi)$  are determined by the cofinality relation,  $x = (x_0, x_1, \dots) \in U_p$  if and only if there is a positive integer  $n = n(x)$  such that  $x_k \in e$  for every  $k \geq n$ . Then  $(\overline{f} \times \overline{f})(G_s(p)) = G_s(p)$ . Let

$$G_{s,n}(p) = \{(x, y) \in G_{s,n} \mid x, y \in U_p\} = \{(x, y) \in G_s(p) \mid f^n(x_0) = f^n(y_0)\}.$$

Then  $G_{s,n}(p)$  is a compact open subset of  $G_s(p)$ , and  $G_{s,n}(p)^0 = G_s(p)^0 = g(U_p)$ . Since  $G_s(p)$  is  $r$ -discrete, the range maps  $r: G_s(p) \rightarrow G_s(p)^0$  and  $r_n = r|_{G_{s,n}(p)}$  are local homeomorphisms. Hence the Haar system of  $G_s(p)$  restricted to  $G_{s,n}(p)$  gives a Haar system for each  $G_{s,n}(p)$ . Then we can express  $C^*(G_s(p))$  as an inductive limit

$$C^*(G_{s,1}(p)) \rightarrow C^*(G_{s,2}(p)) \rightarrow \cdots \rightarrow C^*(G_{s,n}(p)) \rightarrow \cdots$$

**Unstable equivalence algebras.** Suppose that  $(\overline{X}, \overline{f})$  is an orientable solenoid and that  $\phi$  is the flow on  $\overline{X}$  given in theorem 2.5. Then there exists a cross section with return time map  $(K, r)$  such that  $\overline{X}$  is the suspension space of  $(K, r)$ .

**Lemma 3.5** ([19, p.59]). *The  $C^*$ -algebra of  $(\overline{X}, \mathbb{R}, \phi)$  is isomorphic to  $C(\overline{X}) \times_{\phi} \mathbb{R}$ .*

**Proposition 3.6** ([12, 17]). *Suppose that  $(\overline{X}, \overline{f})$  is an orientable solenoid, and that  $(Z, r)$  is a cross section with the return time map of the flow  $\phi$ . Then*

- (1)  $U(\overline{X}, \overline{f}) \simeq C(\overline{X}) \times_{\phi} \mathbb{R}$  and
- (2)  $C(\overline{X}) \times_{\phi} \mathbb{R}$  is strongly Morita equivalent to  $C(K) \times_r \mathbb{Z}$ .

*Proof.* (1). Suppose  $x = (x_0, x_1, \dots), y = (y_0, y_1, \dots) \in \overline{X}$  and  $(x, y) \in G_u$ . Then  $d(\overline{f}^n(x), \overline{f}^n(y)) \rightarrow 0$  as  $n \rightarrow -\infty$  implies  $d_0(x_n, y_n) \rightarrow 0$  as  $n \rightarrow \infty$  and that there exists a  $t \in \mathbb{R}$  such that  $y = \phi_t(x)$ . Let  $\alpha: (\overline{X}, \mathbb{R}, \phi) \rightarrow G_u$  be given by  $(x, t) \mapsto (x, \phi_t(x))$ . Then it is not difficult to see that  $\alpha$  is an isomorphism. Therefore  $U(\overline{X}, \overline{f})$  is isomorphic to  $C(\overline{X}) \times_{\phi} \mathbb{R}$  by lemma 3.5. And by the same argument  $C(K) \times_r \mathbb{Z}$  is isomorphic to the groupoid  $C^*$ -algebra of  $(K, \mathbb{Z}, r)$ .

(2). Since  $\overline{X}$  is the suspension of  $(K, r)$ , for every  $x \in \overline{X}$  there exist unique  $z_x \in K$  and  $\tau_x \in [0, 1)$  such that  $x = \phi_{\tau_x}(z_x)$ . Define  $I = \{(x, n - \tau_x) \mid x \in \overline{X}, n \in \mathbb{Z}\}$ , and let  $\mathcal{C}(I)$  be the completion of  $C_c(I)$ . Then by the Theorem in [17, §4.a]  $\mathcal{C}(I)$  is a  $C(\overline{X}) \times_{\phi} \mathbb{R}$ - $C(K) \times_r \mathbb{Z}$  imprimitivity bimodule. For completeness, we write down the module structures and the inner products.

*Module structures.* Suppose that  $\alpha \in C_c(I)$ ,  $g \in C_c(\overline{X}, \mathbb{R}, \phi)$  and  $h \in C_c(K, \mathbb{Z}, r)$ . Then

$$\begin{aligned} (g \cdot \alpha)(x, n - \tau_x) &= \int g(x, t) \cdot \alpha(\phi_t(x), n - \tau_x - t) d\mu^{[x]}(t) \text{ and} \\ (\alpha \cdot h)(x, n - \tau_x) &= \sum_m \alpha(x, m - \tau_x) \cdot h(r^m(z_x), n - m) \end{aligned}$$

give that  $\mathcal{C}(I)$  is a left  $C(\overline{X}) \times_{\phi} \mathbb{R}$  and right  $C(K) \times_r \mathbb{Z}$  bimodule with  $(\tilde{g} \cdot \tilde{\alpha}) \cdot \tilde{h} = \tilde{g} \cdot (\tilde{\alpha} \cdot \tilde{h})$  for every  $\tilde{\alpha} \in \mathcal{C}(I)$ ,  $\tilde{g} \in C(\overline{X}) \times_{\phi} \mathbb{R}$  and  $\tilde{h} \in C(K) \times_r \mathbb{Z}$ .

*Inner products.* Define  $\langle \cdot, \cdot \rangle_L: C_c(I) \times C_c(I) \rightarrow C_c(\overline{X}, \mathbb{R}, \phi)$  and  $\langle \cdot, \cdot \rangle_R: C(I) \times C(I) \rightarrow C_c(K, \mathbb{Z}, r)$  by

$$\begin{aligned} \langle \alpha, \beta \rangle_L(x, t) &= \sum \alpha(x, m - \tau_x) \cdot \overline{\beta(x, m - \tau_x)} \text{ and} \\ \langle \alpha, \beta \rangle_R(z, k) &= \int \overline{\alpha(\phi_t(z), k - t)} \cdot \beta(\phi_t(z), k - t) d\mu^{[\phi_t(z)]}(t). \end{aligned}$$

□

Then we have the following corollary from propositions 2.6.

**Corollary 3.7** ([5, 26]). (1)  $U(\overline{X}, \overline{f})$  is a simple  $C^*$ -algebra.

(2)  $K_1(U(\overline{X}, \overline{f})) = \mathbb{Z}$ .

(3)  $K_0(U(\overline{X}, \overline{f}))$  is order isomorphic to  $\Delta_M$  where  $M$  is the adjacency matrix of  $(\overline{X}, \overline{f})$ .

Recall that the flow  $\phi$  on  $\overline{X}$  is uniquely ergodic without rest point (theorem 2.5). So  $C(\overline{X}) \times_{\phi} \mathbb{R}$  has the unique trace  $\tau_{\mu}$  induced by the Williams measure  $\mu$  ([22, 3.3.10]). Thus  $\tau_{\mu}^*$ , the induced state on  $K_0(C(\overline{X}) \times_{\phi} \mathbb{R})$ , is the unique state.

**Proposition 3.8.** *Suppose that  $(\overline{X}, \overline{f})$  is a 1-solenoid and that  $M$  is the corresponding adjacency matrix with the normalized Perron eigenvector  $\mathbf{v} = (v_1, \dots, v_n)$ . Then*

$$\tau_\mu^* (K_0(U(\overline{X}, \overline{f})), K_0(U(\overline{X}, \overline{f}))_+) = \langle (\Delta_M, \Delta_M^+, \mathbf{v}) \rangle.$$

*Proof.* Suppose that  $\mathcal{E}_k = \mathcal{E}$  is the edge set of the  $k$ th coordinate space of  $\overline{X}$ . Then by proposition 2.6

$$(K_0(U(\overline{X}, \overline{f})), K_0(U(\overline{X}, \overline{f}))_+) \cong \left( \varinjlim C(\mathcal{E}_k, \mathbb{Z}), \varinjlim C_+(\mathcal{E}_k, \mathbb{Z}) \right) \cong (\Delta_M, \Delta_M^+).$$

For  $g \in C(\mathcal{E}_k, \mathbb{Z})$ ,  $x = (x_0, \dots, x_k, \dots) \in \overline{X}$  with  $x_k = e^{2\pi i s} \in e_i \in \mathcal{E}_k$  and the canonical projection to the  $k$ th coordinate space  $\pi_k: \overline{X} \rightarrow X$ , define  $g_k \in C(X_k, S^1)$  and  $\tilde{g} \in C(\overline{X}, S^1)$  by

$$g_k: x_k \mapsto \exp(2\pi i g(e_i)s) \text{ and } \tilde{g}: x \rightarrow g_k \circ \pi_k(x).$$

Then every  $\tilde{g}$  is a unitary element in  $C(\overline{X})$ , and  $K_0(U(\overline{X}, \overline{f})) \cong K_1(C(\overline{X}))$  is generated by  $\tilde{g}$ . If we denote  $g$  as  $(g(e_1), \dots, g(e_n))$ , then by Theorem 2.2 of [13]

$$\begin{aligned} \tau_\mu^*(\tilde{g}) &= \frac{1}{2\pi i} \int_{\overline{X}} \frac{\tilde{g}'}{\tilde{g}} d\mu = \int_{X_k} g' d\mu_0 = \sum_{i=1}^n g(e_i) \mu_0(e_i) = \sum_{i=1}^n g(e_i) v_i \\ &= \langle (g(e_1), \dots, g(e_n)), \mathbf{v} \rangle. \end{aligned}$$

□

The above proposition refines Theorem 2.2 of [13] that

$$\tau_\mu^*(K_0(C(\overline{X}) \times_\phi \mathbb{R})) = \langle A_\mu, \check{H}^1(\overline{X}) \rangle.$$

**Corollary 3.9** ([2]). *If  $p$  and  $q$  are projections in  $M_\infty(C(\overline{X}) \times_\phi \mathbb{R})$  such that  $\tau_\mu(p) < \tau_\mu(q)$ , then  $p$  is equivalent to a subprojection of  $q$ .*

**Lemma 3.10** ([15]).  *$C(K) \times_r \mathbb{Z}$  has real rank zero and topological stable rank one.*

Since  $C(\overline{X}) \times_\phi \mathbb{R}$  and  $C(K) \times_r \mathbb{Z}$  are separable algebras, they have strictly positive elements. So strong Morita equivalence of  $C(\overline{X}) \times_\phi \mathbb{R}$  and  $C(K) \times_r \mathbb{Z}$  implies that they are stably isomorphic, i.e.,  $\{C(\overline{X}) \times_\phi \mathbb{R}\} \otimes \mathcal{K}$  is  $*$ -isomorphic to  $\{C(K) \times_r \mathbb{Z}\} \otimes \mathcal{K}$  where  $\mathcal{K}$  is the algebra of compact operators on a separable Hilbert space. Therefore we have the following proposition.

**Proposition 3.11.**  *$U(\overline{X}, \overline{f})$  has real rank zero and topological stable rank one.*

#### 4. RUELLE ALGEBRAS FOR SOLENOIDS

We compute  $K$ -groups of Ruelle algebras for 1-solenoids to show that they are  $*$ -isomorphic.

**Unstable equivalence Ruelle algebras.** Suppose that  $(\overline{X}, \overline{f})$  is an oriented 1-solenoid and that  $G_u \simeq (\overline{X}, \mathbb{R}, \phi)$  is the unstable equivalence groupoid on  $\overline{X}$ . Recall that for  $x, y \in \overline{X}$  such that  $y = \phi_t(x)$ ,  $t \in \mathbb{R}$ , we have  $\overline{f}^{-1}(y) = \phi_{t\lambda^{-1}} \circ \overline{f}^{-1}(x)$ .

**Definition 4.1** ([17, §4]). Let  $\alpha_u$  be an automorphism on  $U(\overline{X}, \overline{f})$  defined by

$$\alpha_u(g)(x, t) = \lambda^{-1} g(\overline{f}^{-1}(x), t\lambda^{-1}) \text{ for } g \in C_c(\overline{X}, \mathbb{R}, \phi) \text{ and } (x, t) \in (\overline{X}, \mathbb{R}).$$

The *unstable equivalence Ruelle algebra*  $R_u(\overline{X}, \overline{f})$  is the crossed product

$$R_u(\overline{X}, \overline{f}) = U(\overline{X}, \overline{f}) \times_{\alpha_u} \mathbb{Z} = (C(\overline{X}) \times_{\phi} \mathbb{R}) \times_{\alpha_u} \mathbb{Z}.$$

*Remarks 4.2.* (1) Let  $A$  be an  $n \times n$  integer matrix and  $\Delta_A$  the dimension group of  $A$ . The *dimension group automorphism*  $\delta_A$  of  $A$  is the restriction of  $A$  to  $A$  so that  $\delta_A(\mathbf{v}) = A\mathbf{v}$  ([10, 7.5.1]). Then  $\Delta_A/\text{Im}(Id - \delta_A)$  is isomorphic to  $\mathbb{Z}^n/(Id - A)\mathbb{Z}^n$ .

(2) For  $g \in C(\mathcal{E}_k, \mathbb{Z})$ , let  $g_k \in C(X_k, S^1)$  be as in the proof of proposition 3.8. The wrapping rule  $\tilde{f}: \mathcal{E}_{k+1} \rightarrow \mathcal{E}_k$  induces a map  $f^*: C(\mathcal{E}_k, \mathbb{Z}) \rightarrow C(\mathcal{E}_{k+1}, \mathbb{Z})$  by  $g \mapsto g \circ \tilde{f}$  where  $(g \circ \tilde{f})(e) = \sum_{i=1}^j g(e_i)$  such that  $\tilde{f}(e) = e_1 \cdots e_j$ . Then  $g_k \circ f \circ \pi_k$  is homotopic to  $(g \circ f^*)_{k+1} \circ \pi_{k+1}$  ([26, 3.6]).

**Proposition 4.3.** Suppose that  $(\overline{X}, \overline{f})$  is a 1-solenoid with the adjacency matrix  $M$  and corresponding dimension group automorphism  $\delta_M$ . Then

$$K_0(R_u(\overline{X}, \overline{f})) \cong \mathbb{Z} \oplus \{\Delta_M/\text{Im}(Id - \delta_M)\} \text{ and } K_1(R_u(\overline{X}, \overline{f})) \cong \mathbb{Z} \oplus \text{Ker}(Id - \delta_M).$$

*Proof.* We have the following Pimsner-Voiculescu exact sequence.

$$\begin{array}{ccccc} K_0(U(\overline{X}, \overline{f})) & \xrightarrow{1-\alpha_{u*}} & K_0(U(\overline{X}, \overline{f})) & \xrightarrow{\iota_*} & K_0(R_u(\overline{X}, \overline{f})) \\ \uparrow & & & & \downarrow \\ K_1(R_u(\overline{X}, \overline{f})) & \xleftarrow{\iota_*} & K_1(U(\overline{X}, \overline{f})) & \xleftarrow{1-\alpha_{u*}} & K_1(U(\overline{X}, \overline{f})) \end{array}$$

We consider  $\alpha_{u*}: K_0(U(\overline{X}, \overline{f})) = K_0(C(\overline{X}) \times_{\phi} \mathbb{R}) \rightarrow K_0(C(\overline{X}) \times_{\phi} \mathbb{R})$  as the automorphism  $\hat{\alpha}_{u*}: K_1(C(\overline{X})) \rightarrow K_1(C(\overline{X}))$  given by the Thom isomorphism of Connes. Define  $\beta: C(\overline{X}) \rightarrow C(\overline{X})$  by  $h \mapsto h \circ \overline{f}^{-1}$  for  $h \in C(\overline{X})$ . Then the induced automorphism  $\beta_*: K_1(C(\overline{X})) \rightarrow K_1(C(\overline{X}))$  is the required isomorphism.

For  $g \in C(\mathcal{E}_k, \mathbb{Z})$ , let  $\tilde{g} \in C(\overline{X}, S^1)$  be the induced unitary element as in the proof of proposition 3.8. Then  $\beta^{-1}(\tilde{g}) = \tilde{g} \circ \overline{f} = g_k \circ \pi_k \circ \overline{f} = g_k \circ f \circ \pi_k$  is homotopic to  $(g \circ f^*)_{k+1} \circ \pi_{k+1}$ . Hence if we denote  $g$  as  $(g(e_1), \dots, g(e_n)) \in \mathbb{Z}^n$ , then  $g \circ f^*$  is given by  $Mg$  and the induced automorphism  $\beta_*^{-1}: K_1(C(\overline{X})) \rightarrow K_1(C(\overline{X}))$  is the dimension group automorphism  $\delta_M$  of the adjacency matrix  $M$ . Therefore  $\beta_*$  is the inverse of  $\delta_M$ , and  $1 - \alpha_{u*}: K_0(U(\overline{X}, \overline{f})) \rightarrow K_0(U(\overline{X}, \overline{f}))$  is the same as  $Id - \delta_M^{-1}: \Delta_M \rightarrow \Delta_M$ .

Since  $K_1(U(\overline{X}, \overline{f}))$  is isomorphic to  $\mathbb{Z}$ ,  $\alpha_{u*}: \mathbb{Z} \rightarrow \mathbb{Z}$  is trivially the identity map. Thus the six-term exact sequence is divided into the following two short exact sequences;

$$0 \rightarrow \Delta_M/\text{Im}(Id - \delta_M^{-1}) \longrightarrow K_0(R_u(\overline{X}, \overline{f})) \longrightarrow \mathbb{Z} \rightarrow 0$$

and

$$0 \rightarrow \mathbb{Z} \longrightarrow K_1(R_u(\overline{X}, \overline{f})) \longrightarrow \text{Ker}(Id - \delta_M^{-1}) \rightarrow 0.$$

Therefore we conclude that

$$\begin{aligned} K_0(R_u(\overline{X}, \overline{f})) &\cong \mathbb{Z} \oplus \{\Delta_M/\text{Im}(Id - \delta_M^{-1})\} = \mathbb{Z} \oplus \{\Delta_M/\text{Im}(Id - \delta_M)\} \text{ and} \\ K_1(R_u(\overline{X}, \overline{f})) &\cong \mathbb{Z} \oplus \text{Ker}(Id - \delta_M^{-1}) = \mathbb{Z} \oplus \text{Ker}(Id - \delta_M). \end{aligned}$$

□

**Examples 4.4.** (1). Suppose that  $X$  is the unit circle and that  $f: X \rightarrow X$  is given by  $z \mapsto z^n$ ,  $n \geq 2$ . Then the adjacency matrix is  $(n)$ ,  $K_0(U(\overline{X}, \overline{f})) = \mathbb{Z}[\frac{1}{n}]$  and  $K_1(U(\overline{X}, \overline{f})) = \mathbb{Z}$ . Since  $\delta_{(n)}^{-1}$  is multiplication by  $\frac{1}{n}$ , we have  $K_0(R_u(\overline{X}, \overline{f})) = \mathbb{Z} \oplus \mathbb{Z}_{n-1}$  and  $K_1(R_u(\overline{X}, \overline{f})) = \mathbb{Z}$ . See [3, 9] for details.

(2). Suppose that  $Y$  is a wedge of two circles  $a$  and  $b$  and that  $g: Y \rightarrow Y$  is given by  $a \mapsto aab$  and  $b \mapsto ab$ . Then the adjacency matrix is  $M = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ . So  $K_0(U(\overline{Y}, \overline{g})) = \mathbb{Z} \oplus \mathbb{Z}$  and  $K_1(U(\overline{Y}, \overline{g})) = \mathbb{Z}$ . Since  $1 - \alpha_{u*}: \mathbb{Z} \oplus \mathbb{Z} \rightarrow \mathbb{Z} \oplus \mathbb{Z}$  is an isomorphism, we obtain  $K_0(R_u(\overline{Y}, \overline{g})) = K_1(R_u(\overline{Y}, \overline{g})) = \mathbb{Z}$ .

**Stable equivalence Ruelle algebras.** We use  $K$ -theoretic duality of the Ruelle algebras and the Universal Coefficient Theorem to compute  $K$ -groups of  $R_s(\overline{X}, \overline{f})$ .

*Remark 4.5* ([20]). Let  $\mathcal{N}$  be the category of separable nuclear  $C^*$ -algebras which contains the separable Type I  $C^*$ -algebras and is closed under strong Morita equivalence, inductive limits, extensions, and crossed products by  $\mathbb{Z}$  and by  $\mathbb{R}$ . Then it is not difficult to verify that unstable and stable equivalence Ruelle algebras of 1-solenoids are contained in  $\mathcal{N}$ .

**Proposition 4.6** ([17, 5.c]). *Suppose that  $(\overline{X}, \overline{f})$  is a 1-solenoid. Then  $R_s(\overline{X}, \overline{f})$  is dual to  $R_u(\overline{X}, \overline{f})$  so that  $K_*(R_s(\overline{X}, \overline{f}))$  is isomorphic to  $K^{*+1}(R_u(\overline{X}, \overline{f}))$ .*

**Proposition 4.7** ([20, 1.19]). *Suppose that  $(\overline{X}, \overline{f})$  is a 1-solenoid. Then there are short exact sequences*

$$\begin{aligned} 0 \rightarrow \text{Ext}_{\mathbb{Z}}^1(K_0(R_u(\overline{X}, \overline{f})), \mathbb{Z}) &\rightarrow K^1(R_u(\overline{X}, \overline{f})) \rightarrow \text{Hom}(K_1(R_u(\overline{X}, \overline{f})), \mathbb{Z}) \rightarrow 0 \\ 0 \rightarrow \text{Ext}_{\mathbb{Z}}^1(K_1(R_u(\overline{X}, \overline{f})), \mathbb{Z}) &\rightarrow K^0(R_u(\overline{X}, \overline{f})) \rightarrow \text{Hom}(K_0(R_u(\overline{X}, \overline{f})), \mathbb{Z}) \rightarrow 0 \end{aligned}$$

Hence  $K$ -groups of the stable equivalence Ruelle algebra are determined by  $\text{Ext}$ - and  $\text{Hom}$ -groups of  $K_*(R_u(\overline{X}, \overline{f}))$ . Transform  $Id - M$  to the Smith form

$$\begin{pmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{pmatrix}$$

where  $d_i \geq 0$  and  $d_i$  divides  $d_{i+1}$  ([10, §7.4]). Then  $\Delta_M / \text{Im}(Id - \delta_M)$  is isomorphic to  $\oplus_{i=1}^n \mathbb{Z}/d_i\mathbb{Z}$ , and the dimension of  $\text{Ker}(Id - \delta_M)$  is equal to the number of zeros in the diagonal of the Smith form. Suppose  $d_1 = \cdots = d_m = 0$  and  $d_{m+1} \neq 0$ . Then we have

$$\begin{aligned} \text{Ext}_{\mathbb{Z}}^1(K_0(R_u(\overline{X}, \overline{f})), \mathbb{Z}) &= \text{Ext}_{\mathbb{Z}}^1(\mathbb{Z}_{d_{m+1}} \oplus \cdots \oplus \mathbb{Z}_{d_n}, \mathbb{Z}) = \mathbb{Z}_{d_{m+1}} \oplus \cdots \oplus \mathbb{Z}_{d_n} \text{ and} \\ \text{Hom}(K_1(R_u(\overline{X}, \overline{f})), \mathbb{Z}) &= \mathbb{Z}^{m+1}. \end{aligned}$$

Hence we have

$$\begin{aligned} K^1(R_u(\overline{X}, \overline{f})) &\cong \text{Hom}(K_1(R_u(\overline{X}, \overline{f})), \mathbb{Z}) \oplus \text{Ext}_{\mathbb{Z}}^1(K_0(R_u(\overline{X}, \overline{f})), \mathbb{Z}) \\ &= \mathbb{Z} \oplus \mathbb{Z}^m \oplus \mathbb{Z}_{d_{m+1}} \oplus \cdots \oplus \mathbb{Z}_{d_n} \\ &\cong \mathbb{Z} \oplus \{\Delta_M / \text{Im}(Id - \delta_M)\}. \end{aligned}$$

Recall that  $K_1(R_u(\overline{X}, \overline{f})) = \mathbb{Z} \oplus \text{Ker}(Id - \delta_M)$  is a torsion-free subgroup of  $\mathbb{Z}^{n+1}$ . Thus we have  $\text{Ext}_{\mathbb{Z}}^1(K_1(R_u(\overline{X}, \overline{f})), \mathbb{Z}) = 0$  and

$$K^0(R_u(\overline{X}, \overline{f})) \cong \text{Hom}(K_0(R_u(\overline{X}, \overline{f})), \mathbb{Z}).$$

Then  $K_0(R_u(\overline{X}, \overline{f})) \cong \mathbb{Z} \oplus_{i=1}^n \mathbb{Z}/d_i\mathbb{Z}$  implies

$$\text{Hom}(K_0(R_u(\overline{X}, \overline{f})), \mathbb{Z}) \cong \text{Hom}(\mathbb{Z} \oplus_{i=1}^n \mathbb{Z}/d_i\mathbb{Z}, \mathbb{Z}) \cong \mathbb{Z} \oplus_{i=1}^m \mathbb{Z} \cong \mathbb{Z} \oplus \text{Ker}(Id - \delta_M).$$

Therefore we conclude that:

**Proposition 4.8.** *Suppose that  $(\overline{X}, \overline{f})$  is a 1-solenoid. Then*

$$K_0(R_s(\overline{X}, \overline{f})) \cong \mathbb{Z} \oplus \{\Delta_M / \text{Im}(Id - \delta_M)\} \text{ and } K_1(R_s(\overline{X}, \overline{f})) \cong \mathbb{Z} \oplus \text{Ker}(Id - \delta_M).$$

*Remark 4.9.* The isomorphisms in proposition 4.8 are *unnatural* as the short exact sequences in the Universal Coefficient Theorem split unnaturally.

Recall that the unstable and stable equivalence Ruelle algebras of a 1-solenoid are nuclear, purely infinite, separable, simple and stable  $C^*$ -algebras (proposition 3.2). Then the classification theorem of Kirchberg-Phillips implies the following proposition.

**Proposition 4.10.**  *$R_u(\overline{X}, \overline{f})$  is  $*$ -isomorphic to  $R_s(\overline{X}, \overline{f})$ .*

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## REFERENCES

1. J. Aarts and M. Martens, *Flows on one-dimensional spaces*, Fund. Math. **131** (1988), 53-67.
2. B. Blackadar, *Comparison theory for simple  $C^*$ -algebras*, Operator algebras and applications, D. E. Evans and M. Takesaki (eds.), LMS lecture Notes Series **135** (1988), 21-54.
3. V. Deaconu, *Groupoids associated with endomorphisms*, Trans. Amer. Math. Soc. **347** (1995), 1779-1786.
4. A. Forrest, *Cohomology of ordered Bratteli diagrams*, Pacific J. Math., to appear.
5. T. Giordano, I. Putnam and C. Skau, *Topological orbit equivalence and  $C^*$ -crossed products*, J. reine angew. Math. **469** (1995), 41-111.
6. R. Herman, I. Putnam and C. Skau, *Ordered Bratteli diagram, dimension groups and topological dynamics*, Intern. J. Math. **3** (1992), 827-864.
7. J. Kaminker, I. Putnam and J. Spielberg, *Operator algebras and hyperbolic dynamics*, Operator algebras and quantum field theory (Rome, 1996), S. Doplicher, R. Longo, J.E. Roberts and L. Zsido (eds.), 525-532, International Press, 1997.
8. E. Kirchberg, *The classification of purely infinite  $C^*$ -algebras using Kasparov's theory*, preprint, 1994.
9. M. Laca and J. Spielberg, *Purely infinite  $C^*$ -algebras from boundary actions of discrete groups*, J. reine angew. Math. **480** (1996), 125-139.
10. D. Lind and B. Marcus, *An introduction to symbolic dynamics and coding*, Cambridge Univ. Press, 1995.
11. B. Marcus, *Unique ergodicity of some flows related to Axiom A diffeomorphisms*, Israel J. Math. **21** (1975), 111-132.
12. P. Muhly, J. Renault and D. Williams, *Equivalence and isomorphism for groupoid  $C^*$ -algebras*, J. Operator Theory **17** (1987), 3-22.
13. J. A. Packer,  *$K$ -theoretic invariants for  $C^*$ -algebras associated to transformations and induced flows*, J. Funct. Anal. **67** (1986), 25-59.
14. N. C. Phillips, *A classification theorem for nuclear purely infinite simple  $C^*$ -algebras*, Doc. Math. **5** (2000), 49-114.

15. I. Putnam, *On the topological stable rank of certain transformation group  $C^*$ -algebras*, Ergod. Th. and Dynam. Sys. **10** (1990), 197-207.
16. I. Putnam,  *$C^*$ -algebras from Smale spaces*, Can. J. Math. **48** (1996), 175-195.
17. I. Putnam, *Hyperbolic systems and generalized Cuntz-Krieger algebras*, Lecture notes from Summer school in Operator algebras, Odense, Denmark, 1996.
18. I. Putnam and J. Spielberg, *The structure of  $C^*$ -algebras associated with hyperbolic dynamical systems*, J. Funct. Anal. **163** (1999), 279-299.
19. J. Renault, *A groupoid approach to  $C^*$ -algebras*, Lecture Notes in Math. **793** (1980) Springer-Verlag.
20. J. Rosenberg and C. Schochet, *The Künneth theorem and the universal coefficient theorem for Kasparov's generalized  $K$ -functor*, Duke Math. J. **55** (1987), 431-474.
21. D. Ruelle, *Noncommutative algebras for hyperbolic diffeomorphisms*, Invent. Math. **93** (1988), 1-13.
22. J. Tomiyama, *Invitation to  $C^*$ -algebras and topological dynamics*, World Scientific Publishing Co. 1987.
23. R. F. Williams, *One-dimensional non-wandering sets*, Topology **6** (1967), 473-487.
24. R. F. Williams, *Classification of 1-dimensional attractors*, Proc. Symp. Pure Math. **14** (1970), 341-361.
25. I. Yi, *Canonical symbolic dynamics for one-dimensional generalized solenoids*, To appear in Trans. Amer. Math. Soc.
26. I. Yi, *Ordered group invariants for one-dimensional spaces*, Submitted for publication.
27. I. Yi, *Orientable double covers and Bratteli-Vershik systems for one-dimensional generalized solenoids*, Preprint.

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